Limits and Derivatives Formulas

1. Limits

Properties

if $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then

$$\lim_{x\to a}\left[f(x)\pm g(x)\right] = l \pm m$$

$$\lim_{x\to a}\left[f(x)\cdot g(x)\right] = l \cdot m$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{l}{m} \text{ where } m \neq 0$$

$$\lim_{x\to a} c\cdot f(x) = c\cdot l$$

$$\lim_{x\to a}\frac{1}{f(x)} = \frac{1}{l} \text{ where } l \neq 0$$

Formulas

$$\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\lim_{x\to\infty}(1+n)^{\frac{1}{n}} = e$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$\lim_{x\to 0}\frac{\tan x}{x} = 1$$

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$$

$$\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$$

$$\lim_{x\to 0}\frac{a^n - 1}{x} = \ln a$$

2. Common Derivatives

Basic Properties and Formulas

$$(cf)' = cf'(x)$$

$$(f \pm g)' = f'(x) + g'(x)$$

Product rule

$$(f\cdot g)' = f'\cdot g + f\cdot g'$$

Quotient rule

$$\left(\frac{f}{g}\right)' = \frac{f'\cdot g - f\cdot g'}{g^2}$$

Power rule

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Chain rule

$$\frac{d}{dx}\left(f\left(g\left(x\right)\right)\right) = f'\left(g(x)\right)g'(x)$$

Common Derivatives

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc \cot x$$

$$\frac{d}{dx}(\cot x) = -\frac{1}{\sin^2 x} = -\csc^2 x$$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

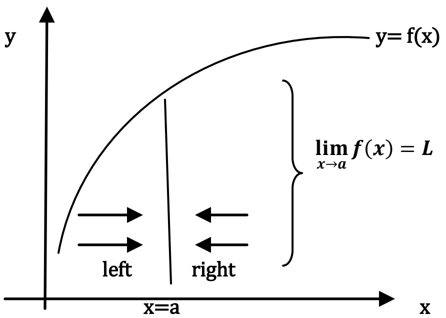
$$\frac{d}{dx}\left(a^x\right) = a^x \ln a$$

$$\frac{d}{dx}\left(e^x\right) = e^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x},\quad x > 0$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x},\ x \neq 0$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a},\ x > 0$$



Relationship between the Limit and one-sided limits

(left (a^-) & right (a^+))

$$\lim_{x\to a^+} f(x) = \lim_{x\to a} f(x) = L = \lim_{x\to a} f(x)$$

Properties

1. $\lim_{x\to a} c = c$
2. $\lim_{x\to a} f(x) = f(a)$

(f(x) =a polynomial or rational func. in the domain of x)

3. $\lim_{x\to a} [c\, f(x)] = c \lim_{x\to a} \mathrm{f(x)}$
4. $\lim_{x\to a} [\mathrm{f(x)} \pm g(x)] = \lim_{x\to a} \mathrm{f(x)} \pm \lim_{x\to a} g(x)$
5. $\lim_{x\to a} [\mathrm{f(x)g(x)}] = \lim_{x\to a} \mathrm{f(x)} \lim_{x\to a} \mathrm{g(x)}$
6. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} \mathrm{f(x)}}{\lim_{x\to a} \mathrm{g(x)}}$ $\quad (\lim_{x\to a} g(x) \neq 0)$
7. $\lim_{x\to a} \mathrm{f(g(x))} = \mathrm{f}\,(\lim_{x\to a} g(x))$
8. $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$
9. $\lim_{x\to a} \sqrt[n]{(f(x))} = \sqrt[n]{\lim_{x\to a} f(x)}$

Indeterminate Forms:

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $(\infty - \infty), (0 \times \infty)$, 1^∞ , 0^0 , ∞^0

(When a limit of rational func. has an indeterminate form,
Simplify the func. by common factors between numerator and denominator.)

$f(x) \le g(x) \ \rightarrow \lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$

Squeeze Theorem

$$f(x) \le g(x) \le h(x) \Rightarrow \lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$
$$\Rightarrow \lim_{x\to a} g(x) = L$$

Absolute function

f(x) = $|x - c|$

1. -(x-c) if x< c
2. 0 if x = c
3. x-c if x> c

Prove Continuous at x = a of f(x)

1. f(a) exists. f(a) is defined at x=a
2. $\lim_{x\to a} f(x)$ *exists.* $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$
3. $f(a) = \lim_{x\to a} f(x)$ *then* $f(x)$ *is cont. at* $x = a$

Any polynomial is continuous everywhere all x.

Any rational function is continuous where it is defined on its domain.

Basic Limit Evaluations at $\pm\infty$

* $\frac{a}{\infty} = 0 \quad \frac{a}{0} = \infty \quad a \times \infty = \infty \quad 0 \times \infty = 0 \quad (a \neq 0, a < \infty)$

1. $\lim_{x\to\infty} ax = a \lim_{x\to\infty} x = a \times \infty = \infty \quad (a < \infty)$
2. $\lim_{x\to\infty} a = a$
3. $\lim_{x\to\infty} \frac{1}{x^n} = \frac{1}{\infty} = 0$
4. $\lim_{x\to\infty} \mathrm{f}(x) = \pm\infty$; *no* Horizontal asymptotes
5. $\lim_{x\to\infty} [a_n x^n + \ldots + a_1] = \lim_{x\to\infty} a_n x^n$ *find the highest power*

① $\lim_{x\to\infty} \frac{m\, x^a}{n\, x^b} = 0 \quad if\ a < b$

② $\lim_{x\to\infty} \frac{m\, x^a}{n\, x^b} = \frac{m}{n} \quad if\ a = b$

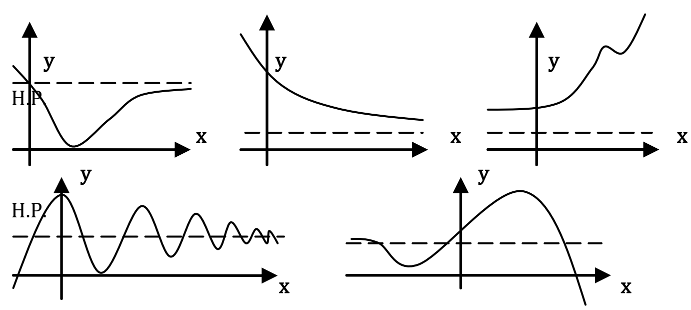
③ $\lim_{x\to\infty} \frac{m\, x^a}{n\, x^b} = \infty \quad if\ a > b$ *no* Horizontal asymptotes

($m, n, a,$ & b *are real numbers*)

6. $\lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to-\infty} e^x = 0$
7. $\lim_{x\to\infty} \ln(x) = \infty \qquad \lim_{x\to-\infty} \ln(x) = -\infty$

Limit at Infinity: Horizontal asymptotes

$\lim_{x\to\infty} \mathrm{f}(x)$; $\frac{\infty}{\infty}$ Indeterminate *form*



Find Vertical Asymptotes ($\lim_{x\to\infty} f(x) = \pm\infty$ *form*)

1. Simplify the func. by common factors between numerator and denominator.
2. Make the denominator =0 for x $\frac{p(x)}{q(x)} \quad p(x) \neq 0$ & $q(x) = 0$
3. x = a is the Vertical Asymptotes.

Limit of Trigonometric Functions

1. $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$
2. $\lim_{\theta\to 0} \sin\theta = 0$
3. $\lim_{\theta\to 0} \cos\theta = 1 \qquad \lim_{\theta\to 0} \frac{1}{\cos\theta} = 1$
4. $\lim_{\theta\to 0} \tan\theta = 0 \qquad \lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1$
5. $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0$
6. $\lim_{\theta\to 0} \sec\theta = 1$
7. $\lim_{\theta\to 0} \frac{\sin a\theta}{\sin b\theta} = \frac{a}{b} \qquad \lim_{\theta\to 0} \frac{\tan a\theta}{\tan b\theta} = \frac{a}{b}$

Definition of the number e

1. $\lim\limits_{h\to 0}\dfrac{e^h - 1}{h} = 1$

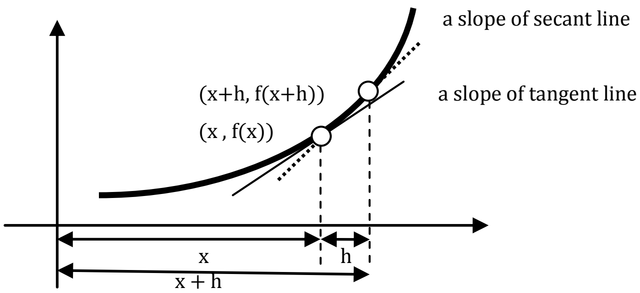
2. $e = \lim\limits_{x\to 0}(x+1)^{\frac{1}{x}} \approx 2.7182818$

$= \lim\limits_{n\to\infty}(1+\frac{1}{n})^n$ (when $\frac{1}{x} = 0$)

Slope; m

1. $m > 0$ positive
2. $m < 0$ negative
3. $m = 0$ Horizontal
4. $m = \infty$ Vertical = no slope

Derivatives and Rates of change



The slope of secant line = $\dfrac{f(x+h) - f(x)}{h}$

= average rate of change or different quotient

The slope of tangent line = m (of f(x) at x=a)

$= \lim\limits_{h\to 0}\dfrac{\boldsymbol{f(x+h) - f(x)}}{\boldsymbol{h}}$

$= \lim\limits_{x\to a}\dfrac{\boldsymbol{f(x) - f(a)}}{\boldsymbol{x - a}}$

= Velocity of f(x) as v

(limit of difference quotient or Derivative of f(x) at x=a)

An Equation of Tangent Line

Use the given f(x) p(x_1, y_1)

1. Find slope m

$f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h) - f(x)}{h}$

2. Find f'$(x_1) = m$
3. y - y_1 = m (x - x_1) --> to make y = ax + b form

Differentiable at x

$f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h) - f(x)}{h}$

Provided the limit exists.

We say that the func. y = f(x) is differentiable at x

Derivatives of y = f (x)

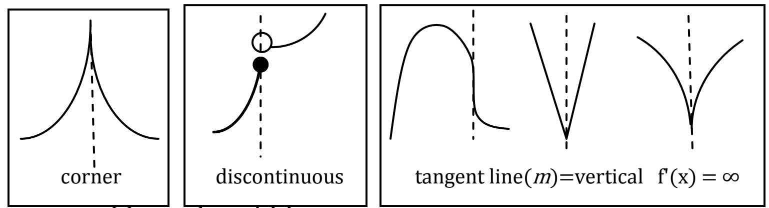
y' = f'(x) = $\frac{dy}{dx} = \frac{d}{dx}f(x)$

Differentiable at a = continuous at a

No differentiable the f(x) could be continuous or not

No Limit, No differentiable

No Differentiable



y = p(x) = a polynomial degree n

Derivatives ; y^n $a\ const.$

$y^{n+1} = 0$ $subsequent\ derivatives$

$ex)$ $y = x^4$, $y''' = 24x$, $y^{4=24}$, $y^5 = 0$

$\boldsymbol{y = 8\cos x}$ $y' = -8\sin x$ $y'' = -8\cos x$ $y''' = 8\sin x$ $\boldsymbol{y^4 = 8\cos x}$

$\boldsymbol{y = 2\sin x}$ $y' = 2\cos x$ $y'' = -2\sin x$ $y''' = -2\cos x$ $\boldsymbol{y^4 = 2\sin x}$

The Linear approximation = a tangent line approximation

The Linearization of at a y = f (x)

$y = m\,(x - x_1) + y_1$ $p\,(x_1, y_1)$ $m = f'(x)$

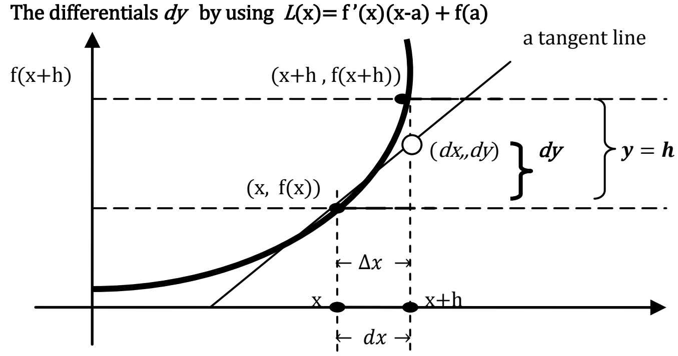
$\boldsymbol{L(x) = f'(a)(x-a) + f(a)}$

The differentials dy by using L(x)= f'(x)(x-a) + f(a)

$L(x) = f'(x)(x-a) + f(a)$

$\boldsymbol{f(x+\Delta x) = f'(x)\Delta x + f(x)}$

The differentials dy by using L(x)= f'(x)(x-a) + f(a)



$\Delta x = x + h - x = h = dx$ $\boldsymbol{\Delta y \neq dy}$

but $\Delta y = f(x+h) - f(x) \approx dy = f'(x)dx = f'(x)\Delta x$

$\dfrac{dy}{dx} = \lim\limits_{h\to 0}\dfrac{f\,(x+h) - f(x)}{h} = \lim\limits_{\Delta x\to 0}\dfrac{f(x-\Delta x) + f(x)}{\Delta x}$

$L(x) = f'(x)(x-a) + f(a)$

$f(x+\Delta x) = f'(x)(x+\Delta x - x) + f(x) = f'(x)\Delta x + f(x)$

$\therefore$ $\boldsymbol{f(x+\Delta x) = f'(x)\Delta x + f(x)}$

$ex)$ sin(0.04) Find Approximate the function

$Let's\ say$ $\sin 0.04 = \sin(x+\Delta x) = \sin(0 + 0.04)$

$f(0) = \sin 0 = 0$ $f'(0) = \cos 0 = 1$

(x of L(x) is 0 because 0.04 is closest to 0)

$L(x) = f(x+\Delta x) = f'(x)\Delta x + f(x)$

$= f'(0)\cdot 0.04 + f(0) = 1\cdot 0.04 + 0 = 0.04$

$ex)$ $e^{-0.015}$ → Let's say $e^{-0.015} = e^{(x+\Delta x)} = e^{(0+(-0.015))}$

$f(0) = e^0 = 1$ $f'(0) = e^0 = 1$

(x of L(x) is 0 because -0.015 is closest to 0)

$L(x) = f(x+\Delta x) = f'(x)\Delta x + f(x)$

$= f'(0)\cdot(-0.015) + f(0) = 1\cdot(-0.015) + 1 = 0.985$

Differentiation Formulas

1. $\frac{d}{dx}\, c = 0$
2. $\frac{d}{dx}\, x = 1$
3. Constant Multiple Rule $\frac{d}{dx}$ c f(x) = c f '(x)
4. Sum & Difference Rule $\frac{d}{dx}$ [f(x) ± g(x)] = f '(x) ± g '(x)
5. Natural Exponential Func. $\frac{d}{dx} e^x = e^x$ $\left(\frac{d}{dx} e^{x+y} = \frac{e^{x+y}}{1-e^{x+y}}\right)$
6. Power Rule $\frac{d}{dx}\, x^n = nx^{n-1}$ (n is any real number)
7. Product Rule $\frac{d}{dx}$[f(x) g(x)] = f ' (x) g(x) + f(x) g '(x)
8. Quotient Rule $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{[g(x)]^2}$
9. Chain Rules $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(\mathrm{g}(x)) = \frac{\mathrm{dy}}{\mathrm{du}} \times \frac{\mathrm{du}}{\mathrm{dx}} = \mathrm{f}'(\mathrm{g}(x)) \cdot \mathrm{g}'(x)$
   1) $\frac{d}{dx}[f(x)]^n = n\,[f(x)]^{n-1} \cdot f'(x)$
   2) $\frac{d}{dx}[e^{f(x)}] = e^{f(x)} \cdot f'(x)$
   3) $\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$
   4) $\frac{d}{dx}$ sin (f(x)) = cos (f(x))· f '(x)
   5) $\frac{d}{dx}$ cos (f(x)) $= -\sin(f(x)) \cdot f'(x)$
   6) $\frac{d}{dx}$ tan (f(x)) = sec^2 (f(x))· f '(x)
   7) $\frac{d}{dx} f(x) \cdot g(x) \cdot h(x)$

$=[f(x) \cdot g(x)]' \cdot h(x) + [\,f(x) \cdot g(x)] \cdot (h'(x))$ (use e^x form to solve)

10. $\frac{d}{dx}\, a^x = \ln a\ a^x$ $a \neq e$
11. $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(\pm x) = \frac{1}{x}$ $x > 0$
12. $\frac{d}{dx}\log_a x = \frac{1}{x\ \ln a}$ $x > 0$
13. $\frac{d}{dx}\, y^n = ny^{n-1}\frac{dy}{dx}$ $\left(\frac{y}{dx}\, y^2 = 2y\frac{dy}{dx}\right)$

Derivatives of Trigonometric Func.

1. $\frac{d}{dx}\sin x = \cos x$
2. $\frac{d}{dx}\cos x = -\sin x$
3. $\frac{d}{dx}\tan x = \sec^2 x = \frac{1}{cos^2 x}$
4. $\frac{d}{dx}\csc x = -\csc x \cot x$
5. $\frac{d}{dx}\sec x = \sec x\ \tan x = \frac{\sin x}{\cos^2 x}$
6. $\frac{d}{dx}\cot x = -csc^2 x = \frac{-1}{\sin^2 x}$
7. $\frac{d}{dx}(\sin x + \cos x) = \cos x - \sin x = \frac{\cot x - 1}{\csc x}$
8. $\frac{d}{dx}\sin(x+y) = \frac{\cos(x+y)}{1 - \cos(x+y)}$
9. $\frac{d}{dx}\cos(x+y) = \frac{-\sin(x+y)}{1 + \sin(x+y)}$

** $\sin^{-1} x \neq \frac{1}{\sin x} = \csc x$ ** $\sin\theta = x \rightarrow \sin^{-1} x = \theta$

Derivatives of Inverse Trigonometric Func.

1. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} = \frac{-1}{\sqrt{x^2-1}}$
2. $\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{x^2-1}}$
3. $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
4. $\frac{d}{dx}\csc^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}$
5. $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$
6. $\frac{d}{dx}\cot^{-1} x == \frac{-1}{1+x^2}$

Hyperbolic Functions

1) $\sinh x = \frac{e^x - e^{-x}}{2}$ $\sinh 0 = 0,$ $\sinh 1 \approx 1.1752$
2) $\cosh x = \frac{e^x + e^{-x}}{2}$ $\cosh 0 = 1,$ $\cosh 1 \approx 1.543$
3) $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}$ $\tanh 0 = 0,$ $\tanh 1 \approx 0.7616$
4) $\operatorname{csch} x = \frac{1}{\sinh x}$
5) $\operatorname{sech} x = \frac{1}{\cosh x}$
6) $\coth x = \frac{1}{\tanh x}$

Derivatives of Hyperbolic Functions

1) $\frac{d}{dx}\sinh x = \cosh x$
2) $\frac{d}{dx}\cosh x = \sinh x$
3) $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$
4) $\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x\ \coth x$
5) $\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x\ \tanh x$
6) $\frac{d}{dx}\coth x = -\operatorname{csch}^2$

$y = \sinh^{-1} x \leftrightarrow \sinh y = x$
$y = \cosh^{-1} x \leftrightarrow \cosh y = x$ $y \geq 0$
$y = \tanh^{-1} x \leftrightarrow \tanh y = x$

Inverse Hyperbolic functions

1) $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$ $(-\infty, \infty)$
2) $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ $[1, \infty)$
3) $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $(-1, 1)$
4) $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$ $(-\infty, 0) \cup (0, \infty)$
5) $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ $(0, 1]$
6) $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $(-\infty, -1) \cup (1, \infty)$

Derivatives of Inverse Hyperbolic functions

1) $\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{x^2+1}}$

2) $\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$

3) $\frac{d}{dx}\tanh^{-1}x = \frac{d}{dx}\coth^{-1}x = \frac{1}{1-x^2}$

4) $\frac{d}{dx}\operatorname{csch}^{-1}x = \frac{-1}{|x|\sqrt{1+x^2}}$

5) $\frac{d}{dx}\operatorname{sech}^{-1}x = \frac{-1}{x\sqrt{1-x^2}}$

Hyperbolic Identities

1) $\cosh x + \sinh x = e^x$

2) $\cosh x - \sinh x = e^{-x}$

3) $\sinh(-x) = -\sinh x$

4) $\cosh(-x) = \cosh x$

5) $cosh^2x - sinh^2x = 1$

6) $tanh^2x + sech^2x = 1$

7) $coth^2x - csch^2x = 1$

8) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

9) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

10) $\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

11) $sinh^2x = \frac{-1 + \cosh 2x}{2}$

12) $cosh^2x = \frac{1 + \cosh 2x}{2}$

13) $\sinh 2x = 2\sinh x \cosh x$

14) $\cosh 2x = cosh^2x + sinh^2x$

15) $\frac{1+\tanh x}{1-\tanh x} = e^{2x}$

16) $(\sinh x + \cosh x)^n = \sinh nx + \cosh nx \quad (n\ is\ an\ real\ number)$

Intermediate Value Theorem

Suppose that f(x) is continuous on [a, b]

Let f(a) < N < f(b) where f(a) ≠ f(b)

Then it exists a number c is belong to (a, b) such that f(c)=N

Extreme Values = Absolute (Global) Values

Max. f(c) ≥ f(x); (the largest) for all x in the domain of f

Min. f(d) ≤ f(x); (the smallest) for all x in the domain of f

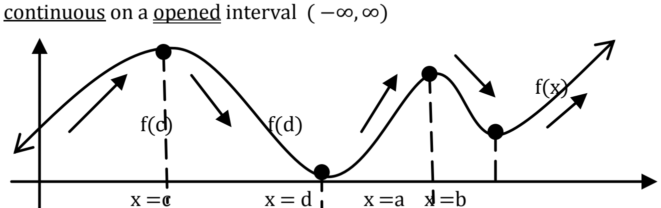
<u>continuous</u> on a <u>closed</u> interval [a,b]

Relative (Local) Values

Max. f(c) ≥ f(x) when x is near c : x= a & c

Min. f(d) ≤ f(x) when x is near d : x= b & d

<u>continuous</u> on a <u>opened</u> interval $(-\infty, \infty)$



a, b, c,& d are critical numbers where f'(x) = 0 and solve for x

Finding <u>Absolute (Global) Max. & Min values</u> (on a closed interval [a, b])

1. f '(x) = 0 → Solve for x c , d... = <u>the Critical Numbers (C.N.)= c, d, ...</u>

2. 1) f(a) & f(b) from [a, b]

   2) f(c) & f(d) from C.N.

3. Max.= the largest value

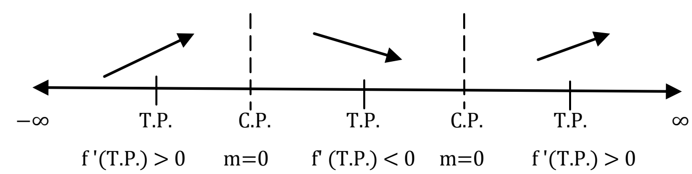
   Min.= the smallest value

Finding <u>Relative (Local) Max. & Min values</u> on an opened interval

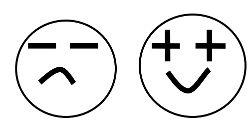
1. f '(x) = 0
   a) Solve for x c , d... = <u>the Critical Numbers (C.N.)</u>
   b) Critical Points: Find f(c), f(d),... --> (c, f(c)) (d, f(d))...
2. Use Arrow diagram
   a) Draw an arrow line $(-\infty, \infty)$
   b) Put C.N, on the line
   c) Choose <u>Testing Points (T.P.)</u> on $(-\infty, C.P.][C.P., C.P.]\ [C.P., \infty)$
3. Increasing interval: f '(T.P.) > 0 ↗

   Decreasing interval: f' (T.P.) < 0 ↘
4. Find Relative (local) Values Max. ( ) at x= ( ),

   Min. ( ) at x= ( )



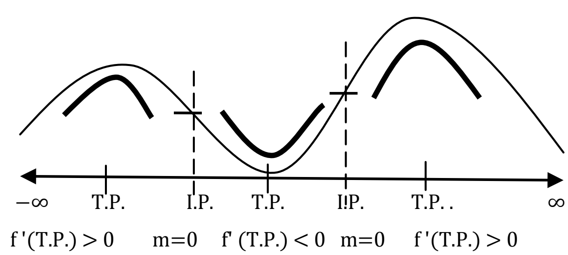
Finding Inflection Points of Concavity Changes



1. Find $f'(x)$
2. $f''(x) = 0$

   Solve for x = <u>Inflection Points I.P.</u>
3. Use Arrow diagram
   a) Draw an arrow line $(-\infty, \infty)$ & Put I.P, on the line
   c) Choose <u>Testing Points (T.P.)</u> on $(-\infty, I.P.][I.P., I.P.]\ [I.P., \infty)$
4. $f''(T.P.) > 0$

   $f''(T.P.) < 0$
5. Inflection Points: where the concavity changes (I.P., f(I.P.))



Some Optimization Problems

1) Suppose that f(x) is continuous on an interval 'I'

   where $f'(x) = 0$ and the x is the <u>only one C.N.</u>

2) If $\boldsymbol{f''(c)} > 0$ Absolute **Min**. at x=c

   If $\boldsymbol{f''(c)} < 0$ Absolute **Max**. at x=c

Simple Apply to Economics' Business

1. Demand Func. = **D(x) = p(x)** (=Price func. that price per unit)
   where x = number of units demanding by consumer at that price 'p.'
   $p \equiv p(x)$

2. **Revenue Func.**
   $\boldsymbol{R(x) = x \cdot p(x)}$ =(sold numbers · selling price)
   Max. of Rev. $= R'(x) = 0$, solve for x
   $\boldsymbol{R'(x)}$ = **Marginal Revenue Func**.

3. **The Profit Func.**
   $\boldsymbol{P(x) = R(x) - C(x)}$
   ; C(x) = Cost Func. (P(x); a capital letter P)
   Max. of Prof .= $P'(x) = 0$, solve for x
   $\boldsymbol{P'(x)}$ = **Marginal Profit Function**

Marginal Analysis

1. Cost Func. $= \boldsymbol{C(x)} = C(x_0)$ by Polynomial (=total cost)

2. **Marginal Cost**

$$\boldsymbol{C'(x) = \lim_{h\to 0}\frac{c\,(x_0 + h) - x\,(x_0)}{h}}$$

It's called 'Marginal Cost' of producing x_0 units.

3. **Actual cost or Actual Revenue** of $c\,(x_0 + 1)$
   If $h = 1$, then $\boldsymbol{C'(x_0) \approx C\,(x_0 + 1) - C\,(x_0)}$

Tip

1. Marginal $\sim = f'(x)$ $\quad ex) R'(x),\ P'(x),\ \&\ C'(x)$

2. To find Max or Min. of Revenue
   1) Find x & $p(x)$
   2) $R(x) = x \cdot p(x)$
   3) $R'(x) = 0$, solve for x, $\quad x = a$, *Q; When Revenue has Max. or Min?*
   4) R(a)= ? (don't forget unit) → *Q; What is Max or Min. of Revenue?*

(Finding Max or Min. of Profit is the same step)

3. To find Actual Revenue from sale of 4th Unit
   1) $R(x) = x \cdot p(x)$
   2) $x_0 = 3$ (to find 4th value)
   3)Find $R(4)$& $R(3)$
   4) $R(4) - R(3) = \$$ ( ) unit

How to solve a Business Calculus' problem

1. Underline all numbers and functions
2. Find what is the main question (ex) Max. of Revenue
3. Find all elements to solve the func. (ex) $R(x) = x \cdot p(x)$
4. Do the next step. (ex) $R'(x) = 0$ solve for x
5. Don't forget unit of the answer. (ex) 40 thousand dollars

L'Hospital's Rule

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

It's good for $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ forms

1. Derivative is continuous 'til it doesn't have the $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ forms.
2. If lim $f(x)$ doesn't have $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ forms, make into the forms

Indeterminate Powers ($0^0, \infty^0, 1^\infty$ forms)

Using Derivatives of Logarithmic Func.

$$\left[\begin{array}{l} u = v \\ \log_a u = \log_a v \end{array}\right. \qquad \left[\begin{array}{l} u = v \\ \ln u = \ln v \end{array}\right. \qquad \left[\begin{array}{l} u = v \\ e^u = e^v \end{array}\right.$$

(Make the same base)

The Intermediate Value Theorem

f(x) is continuous on [a, b] - a closed interval
Let $f(a) < N < f(b)$, where $f(a) \neq f(b)$
Then $\exists$ $'C' \in (a, b)$ — an open interval
such that $f(c) = N$

***** $\exists$ = *There exists*
***** $\in$ = *Belong to*

Rolle's Theorem

1) f(x) is continuous on [a, b]
2) f(x) is differentiable in (a, b)
3) f**(a) = f(b)**
4) Then Then $\exists$ $'C' \in (a, b)$ such that $f'(c) = 0 \quad x = c$

Mean Value Theorem

1) f(x) is continuous on [a, b]
2) f(x) is differentiable in (a, b)
3) **f(a) ≠ f(b)**
4) $\boldsymbol{f'(c) = \dfrac{f(b) - f(a)}{b - a}}$ $\quad x = c$
   $f(b) - f(a) = f'(c)(b - a)$

Newton's Method

1. $f(x) = 0$, $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \quad f'(x) \neq 0$

* x_{n+1}; $(n+1)$approx. of x
  x_n; nth approx. of x

2. Suppose n = 0 , $\quad x_0 - \dfrac{f(x_0)}{f'(x_0)} = x_1$

ex) x_0 is given ,

$$x_0 - \frac{f(x_0)}{f'(x_0)} = x_1$$

$$x_1 - \frac{f(x_1)}{f'(x_1)} = x_2$$

$$x_2 - \frac{f(x_2)}{f'(x_2)} = x_3$$

... (cont.)

*Keep repeating it 'til two numbers are very close each other & then stop.

Antiderivative

$y = f(x)$

suppose $\frac{d}{dx}\,y = \frac{d}{dx}\,f(x) = g(x)$

then f(x) called Antiderivative of g(x) w.r.t. x, then Notation $\int \boldsymbol{g(x)\,dx}$

Properties

$\int f(x) \pm g(x)\,dx = \int f(x)dx \pm \int g(x)$ dx

$\int C\,f(x)dx = C\int f(x)\,dx$ $(C\ is\ constant)$

Basic of Integral

1. $\int \mathrm{k\,d}x = \mathrm{k}\,x + \mathrm{c}$
2. $\int x^{\mathrm{n}}\,\mathrm{d}x = \dfrac{x^{n+1}}{n+1} + \mathrm{C},\quad (n \neq -1)$
3. $\int x^{-1}dx = \int \dfrac{1}{x}\,dx = \ln|x| + C$
4. $\int e^{x}\,dx = e^{x} + C$
5. $\int e^{kx}\,dx = \dfrac{1}{k}e^{kx} + C \qquad (k \neq 0)$
6. $\int a^{x}\,dx = \dfrac{a^{x}}{\ln a} + C$
7. $\int y\,dx = xy - \int y\,dx$

Trigonometric Forms

1. $\int \sin x\,\,dx = -\cos x + C$
2. $\int \cos x\,\,dx = \sin x + C$
3. $\int \sec^2 x\,\,dx = \tan x + C$
4. $\int \csc^2 x\,dx = -\cot x + C$
5. $\int \sec x\,\tan x\,\,dx = \sec x + C$
6. $\int \csc x\,\cot x\,\,dx = -\csc x + C$
7. $\int \tan x\,\,dx = \ln|\sec x| + C = -\ln|\cos x| + C$
8. $\int \cot x\,\,dx = \ln|\sin x| + \mathrm{C} = -\ln|\mathrm{css}\,x| + C$
9. $\int \sec x\,\,dx = \ln|\sec x + \tan x| + C$
10. $\int \csc x\,dx = \ln|\csc x - \cot x| + C$
11. $\int \dfrac{1}{x^2+1}dx = \tan^{-1} x + C$
12. $\int \dfrac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C \qquad (x^2 < 1)$
13. $\int \dfrac{1}{a^2 + x^2}\,\,dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$
14. $\int \dfrac{1}{\sqrt{\mathrm{a}^2 - x^2}}\,dx = \sin^{-1}\left(\dfrac{x}{a}\right) + C \qquad (x^2 < a^2)$
15. $\int \dfrac{1}{x\sqrt{x^2 - a^2}}\,\mathrm{dx} = \dfrac{1}{a}\sec^{-1}\dfrac{x}{a} + C \qquad (x^2 > a^2)$
16. $\int \sin ax\,dx = \dfrac{-\cos ax}{a} + C$

Exponential & Logarithmic Forms

1. $\int xe^{ax}\,\,dx = \dfrac{1}{a^2}\,(ax-1)e^{ax} + C$
2. $\int x^n e^{ax}\,\,dx = \dfrac{1}{a}\,x^n e^{ax} - \dfrac{n}{a}\int x^{n-1}e^{ax}\,dx$
3. $\int e^{ax}\,\sin bx\,\,dx = \dfrac{e^{ax}}{a^2 + b^2}\,(a\,\sin bx - b\,\cos bx) + C$
4. $\int e^{ax}\,\cos bx\,\,dx = \dfrac{e^{ax}}{a^2 + b^2}\,(a\,\cos bx + b\,\sin bx) + C$
5. $\int x^n\,\ln x\,\,dx = \dfrac{x^{n+1}}{(n+1)^2}\,[(n+1)\ln x - 1] + C$
6. $\int \dfrac{1}{x\,\ln x}\,dx = \ln|\ln x| + C$
7. $\int \dfrac{1}{\mathrm{ax+b}}\,\mathrm{d}x = \dfrac{1}{a}\,\ln|\mathrm{a}x + \mathrm{b}| + \mathrm{C} \qquad (a \neq 0)$
8. $\int \dfrac{x}{ax^2 \pm b}\,dx = \dfrac{1}{2a}\ln|ax^2 \pm b| + C \qquad (a \neq 0)$
9. $\int \ln x\,\,\mathrm{d}x = x\,\ln(x) - x + C$
10. $\int \dfrac{dx}{a^2 - x^2} = \dfrac{1}{2a}\,\ln\left|\dfrac{x+a}{x-a}\right| + \mathrm{C}$
11. $\int \dfrac{dx}{x^2 - a^2} = \dfrac{1}{2a}\,\ln\left|\dfrac{x-a}{x+a}\right| + \mathrm{C}$

Hyperbolic Forms

1. $\int \sinh x\,\,dx = \cosh x + C$
2. $\int \cosh x\,\,dx = \sinh x + C$
3. $\int \tanh\,x\,dx = \ln\cosh x + C$
4. $\int \coth x\,\,dx = \ln|\sinh\,x| + C$
5. $\int \mathrm{sech}\,\,x\,dx = \tan^{-1}|\sinh x| + C$
6. $\int \mathrm{csch}\,x\,\,dx = \ln\left|\tanh\dfrac{1}{2}\,x\right| + C$
7. $\int \mathrm{sech}^2\,x\,\,dx = \tanh\,x + C$
8. $\int \mathrm{csch}^2\,x\,\,dx = -\coth\,x + C$
9. $\int \mathrm{sech}\,\,x\,\,\tanh\,\,x\,\,dx = -\mathrm{sech}\,\,x + C$
10. $\int \mathrm{csch}\,x\,\,\coth\,\,x\,\,dx = -\mathrm{csch}\,x + C$

Differentiation Formulas

1. $\frac{d}{dx}\ c = 0$
2. $\frac{d}{dx}\ x\ =\ 1$ $\qquad \frac{d}{dx}\ kx\ =\ k$
3. $\frac{d}{dx}x^n = nx^{n-1}$
4. $\frac{d}{dx}\ e^x = e^x$
5. $\frac{d}{dx}\ a^x =\ \ln a\ \ a^x$ $\qquad a\ \neq e$
6. $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(\pm x) =\ \frac{1}{x}$ $\qquad x > 0$
7. $\frac{d}{dx}\log_a x = \frac{1}{x\ \ \ln a}$ $\qquad x > 0$
8. $\frac{d}{dx}\sin x = \cos x$
9. $\frac{d}{dx}\cos x = -\sin x$
10. $\frac{d}{dx}\tan x = \sec^2 x$
11. $\frac{d}{dx}\csc x = -\csc x \cot x$
12. $\frac{d}{dx}\sec x =\ \sec x\ \tan x$
13. $\frac{d}{dx}\cot x = -\csc^2 x$
14. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
15. $\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
17. $\frac{d}{dx}\csc^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}$
18. $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$
19. $\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$
20. $\frac{d}{dx}\ \sinh x =\ \cosh x$
21. $\frac{d}{dx}\ \cosh x =\ \sinh x$
22. $\frac{d}{dx}\ \tanh x =\ \text{sech}^2 x$
23. $\frac{d}{dx}\ \text{csch}\, x =\ -\text{csch}\, x\ \coth x$
24. $\frac{d}{dx}\ \text{sech}\, x =\ -\text{sech}\, x\ \tanh x$
25. $\frac{d}{dx}\ \coth x =\ -\text{csch}^2$
26. $\frac{d}{dx}\ \sinh^{-1} x = \frac{1}{\sqrt{x^2+\ 1}}$
27. $\frac{d}{dx}\ \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$
28. $\frac{d}{dx}\ \tanh^{-1} x\ =\ \frac{d}{dx}\ \coth^{-1} x\ =\ \frac{1}{1-\ x^2}$
29. $\frac{d}{dx}\ \text{csch}^{-1} x = \frac{-1}{|x|\sqrt{1+x^2}}$
30. $\frac{d}{dx}\ \text{sech}^{-1} x = \frac{-1}{x\sqrt{1-x^2}}$

Antiderivative(Integral) Formulas

$$\int \text{k}\, \text{d}x = \text{k}\, x + \text{c}$$

$$\int x^{\text{n}}\, \text{d}x = \frac{x^{n+1}}{n+1} + \text{C}, \qquad (n \neq -1)$$

$$\int e^x\, dx =\ e^x +\ C \qquad \int e^{kx}\, dx = \frac{1}{k}e^{kx} + C \qquad (k \neq 0)$$

$$** \int a^x\, dx = \frac{a^x}{\ln a} + C$$

$$\int x^{-1} dx = \int \frac{1}{x}\, dx =\ \ln|x| +\ C$$

$$\int \cos x\ \ dx =\ \sin x + C$$

$$\int \sin x\ \ dx =\ -\ \cos x +\ C$$

$$\int \sec^2 x\ \ dx =\ \tan x + C$$

$$\int \csc x\ \cot x\ \ dx =\ -\ \csc x +\ C$$

$$\int \sec x\ \tan x\ \ dx =\ \sec x +\ C$$

$$\int \csc^2 x\ dx =\ -\cot x +\ C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx =\ \sin^{-1} x + C \qquad \int \frac{1}{\sqrt{\text{a}^2-x^2}}\ dx =\ \sin^{-1}\frac{x}{a} +\ C \qquad (x^2 < a^2)$$

$$\int \frac{1}{x^2+1} dx =\ \tan^{-1} x + C \qquad \int \frac{1}{a^2+\ x^2}\ dx = \frac{1}{a}\ \tan^{-1}\frac{x}{a} +\ C \quad (x^2 < a^2)$$

$$\int \frac{1}{x\sqrt{x^2-\ a^2}}\ dx = \frac{1}{a}\ \sec^{-1}\frac{x}{a} + C \qquad (x^2 > a^2)$$

$$\int \cosh x\ \ dx =\ \sinh x +\ C$$

$$\int \sinh x\ \ dx =\ \cosh x +\ C$$

$$\int \text{sech}^2\, x\ \ dx =\ \tanh\ x +\ C$$

$$\int \text{csch}\, x\ \coth x\ \ dx =\ -\text{csch}\, x\ +\ C$$

$$\int \text{sech}\ x\ \tanh\ x\, dx =\ -\ \text{sech}\ x +\ C$$

$$\int \text{csch}^2\, x\ \ dx =\ -\ \coth\ x +\ C$$

Antiderivatives of $f(x)$ = Indefinite Integral

$$\int f(x)\,dx = F(x) \leftrightarrow F'(x) = f(x) = \frac{d}{dx}F(x)$$

$f(x)$is continuous.

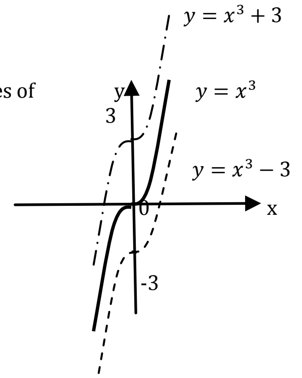
F(x) is called Antiderivative of f on an interval I

if $F'(x) = f(x)$for all x in I.

*Member of the family of Antiderivatives of
$F(x) = y = x^3 + C$

C	$\int 3x^2 dx = x^3 + C$
0	$= x^3 + 0$
3	$= x^3 + 3$
-3	$= x^3 - 3$

(C is an arbitrary constant.)



The Substitution Rule

1. Let $\boldsymbol{g(x) = u}$
2. $g(x)$ is a differentiable func. whose range is an interval I
   f iscontinuous on I
3. $g'(x) = \frac{du}{dx}$ $\rightarrow$ $\boldsymbol{g'(x)dx = du}$ then

$\therefore$ $\int \underline{\boldsymbol{f(g(x))}}\ \underline{\boldsymbol{g'(x)\,dx}} = \int \underline{\boldsymbol{f(u)}}\ \underline{\boldsymbol{du}}$

** $f(g(x)) = f(u)$, $\quad g'(x)dx = du$

$ex) \int e^{-x}\,dx$

1. Let $-x = u$,
2. Derivative both sides $-1 = \dfrac{du}{dx}$
3. $dx = -du$ $\quad then$

$\int e^u - du = -\int e^u\,dx = -e^u + C = -e^{-x} + C$ (because $u = -x$)

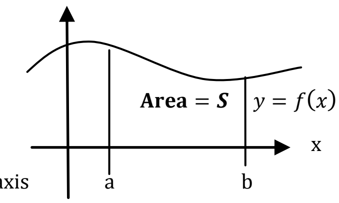
Integral		Substitution	
$\int (3x+4)^{5/2}\,dx$;	$3x + 4 = u$	$3 = \dfrac{du}{dx}$	$dx = \dfrac{1}{3}du$
$\int \dfrac{4}{3-x}\,dx$;	$3 - x = u$	$-1 = \dfrac{du}{dx}$	$dx = -du$
$\int t \cdot e^{2-t^2}\,dt$;	$2 - t^2 = u$	$-2t = \dfrac{du}{dt}$	$t\,dt = \dfrac{-1}{2}du$
$\int t(2+t^2)^3\,dt$;	$2 + t^2 = u$	$2t = \dfrac{du}{dt}$	$t\,dt = \dfrac{1}{2}du$
$\int \dfrac{3}{(2x-5)^4}\,dx$;	$2x - 5 = u$	$2 = \dfrac{du}{dx}$	$dx = \dfrac{1}{2}du$
$\int x^2 e^{-x^3}\,dx$;	$x^3 = u$	$3x^2 = \dfrac{du}{dx}$	$x^2 dx = \dfrac{1}{3}du$
$\int \dfrac{e^t}{e^t+1}\,dt$;	$e^t + 1 = u$	$e^t = \dfrac{du}{dt}$	$e^t\,dt = du$
$\int \dfrac{x+3}{\sqrt[3]{x^2+6x+5}}\,dx$;	$x^2 + 6x + 5 = u$	$2x + 6 = \dfrac{du}{dx}$	$x + 3\,dx = \dfrac{1}{2}du$

$\int x\sqrt{x-1}\,dx = \int u^{1/2}\,x\,dx = \int u^{1/2}\,x\,du = \int u^{1/2}\,(u+1)du$;

$x - 1 = u$, $\quad 1 = \dfrac{du}{dx}$, $\quad dx = du$, $\quad x = u + 1$

Definite Integral

$$\int_a^b \boldsymbol{f(x)\ dx} \qquad a \le x \le b$$



This represents the area
under the curve y=f(x) bounded by x-axis
and the lines x=a and x=b.

1) Left and Right Endpoints

It's hard to find the area of a region with curved sides,

so we use the idea that the slope of tangent line by slopes of secants lines

and the limit of these approximations.

Suppose we divides $\boldsymbol{S}$ into $\boldsymbol{n}$th strips and the area A is between left and

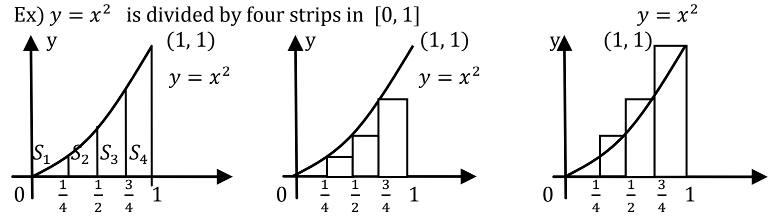
right endpoints of the rectangles. (the width Δx of all strips are same.)

** Right Enfpoints $= \lim_{n\to\infty} R_n$, ** Left Enfpoints $= \lim_{n\to\infty} L_n$

$$\textbf{Area} = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [\boldsymbol{f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x + f(x_n)\Delta x}]$$

$$= \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x]$$

Ex) $y = x^2$ is divided by four strips in [0, 1]



Area=A $\qquad$ Using left end points $\qquad$ Using right end points

$\boldsymbol{\Delta x}$ is the width of each stip $= \dfrac{\text{b} - \text{a}}{\text{n}} = \dfrac{1-0}{4} = \dfrac{1}{4}$

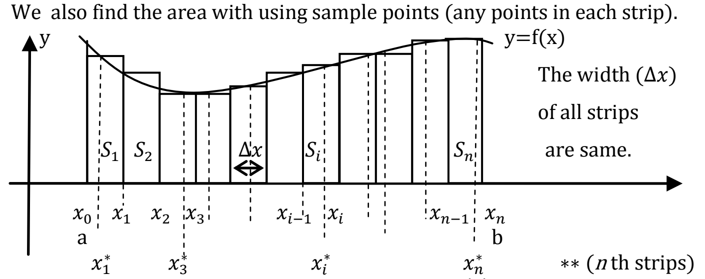
The area using Left Endpoints$= \frac{1}{4}\,[0^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2] = 0.21875$

The area using Right Endpoints$= \frac{1}{4}\,[\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2 + 1^2] = 0.46875$

$\therefore$ $0.21875 <$ the $Area$ of $y = x^2$ in $[0, 1]$ < 0.46875

2) Sample Points (Reimann Integral; Sample points)

We also find the area with using sample points (any points in each strip).



The height if the i th rectangle to be the value of $f(x)$ at any number x_i^*

in the i th subinterval $[x_{i-1}, x_i]$ $\qquad$ ($x_i = a + i\,\Delta x$)

$$\textbf{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \boldsymbol{f(x_i^*)\ \Delta x} = \lim_{n\to\infty} [\boldsymbol{f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x}]$$

$$= \int_a^b \boldsymbol{f(x)\ dx = F(b) - F(a)}$$

$$= [F(x) + C]_a^b = [F(b) + C] - [F(a) + C]$$

3) Midpoint Rule

$$\int_a^b \boldsymbol{f(x)\ dx} \approx \sum_{i=1}^{n} \boldsymbol{f(\bar{x}_i)\ \Delta x} = \boldsymbol{\Delta x\,[f(\bar{x}_1) + \cdots + f(\bar{x}_n)]}$$

where $\Delta x = \dfrac{b-a}{n}$ and $\boldsymbol{\bar{x}_i = \dfrac{1}{2}(x_{i-1} + x_i)}$ = midpoints of $[x_{i-1}, x_i]$

Ex) Use theRight endpoint & *Midpoint Rule* with $n = 3$

to approximate $\int_1^2 \frac{1}{x}\, dx$

1- Using Right Endpoint

$\Delta x = \frac{b-a}{n} = \frac{2-1}{3} = \frac{1}{3}$

$x_0 = a = 1$

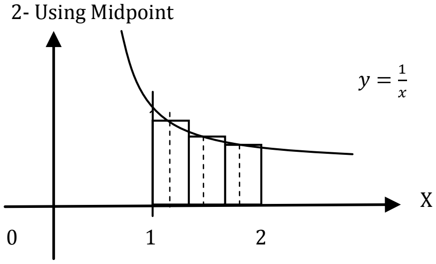
$x_1 = a + i\Delta x = 1 + 1 \cdot \frac{1}{3} = \frac{4}{3}$

$x_2 = x_1 + \Delta x = \frac{4}{3} + \frac{1}{3} = \frac{5}{3}$

$x_3 = x_2 + \Delta x = \frac{5}{3} + \frac{1}{3} = 2$

$\Delta x[f(x_1) + f(x_1) + f(x_1)] = \frac{1}{3}\left(\frac{1}{4/3} + \frac{1}{5/3} + \frac{1}{2}\right) \approx 0.6167$

2- Using Midpoint



$\sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x = \sum_{n=1}^{3} \frac{1}{\bar{x}_i}\,\Delta x = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3)]$

$x_1 = a + i\Delta x = 1 + 1 \cdot \frac{1}{3} = \frac{4}{3}$

$\bar{x}_1 = \frac{1}{2}(x_{i-1} + x_i) = \frac{1}{2}(x_0 + x_1) = \frac{1}{2}\left(1 + \frac{4}{3}\right) = \frac{7}{6}$

$\bar{x}_2 = \bar{x}_1 + \Delta x = \frac{7}{6} + \frac{1}{3} = \frac{3}{2}$

$\bar{x}_3 = \bar{x}_2 + \Delta x = \frac{3}{2} + \frac{1}{3} = \frac{11}{6}$

$\Delta x[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3)] = \frac{1}{3}\left(\frac{1}{7/6} + \frac{1}{3/2} + \frac{1}{11/6}\right) \approx 0.6898$

3. $\int_1^2 \frac{1}{x}\, dx = [\ln|x|]_1^2 = \ln 2 - \ln 1 \approx 0.6932$

$Ex)\ \int_0^2 3x^2 - 2\ dx = \left[3\frac{x^3}{3} - 2x\right]_0^2 = (2^3 - 4) - 0 = 4$

$Ex)\ \int_0^{\pi/2} \sin x\ dx = [-\cos x]_0^{\frac{\pi}{2}} = -\left[\cos\frac{\pi}{2} - \cos 0\right] = -(0-1) = 1$

$Ex)\ \int_{-1}^{2} (x - 2|x|)dx = \int_{-1}^{2} x\ dx - 2\int_{-1}^{2} |x|\, dx$

$= \int_{-1}^{2} x\ dx - 2\left(\int_{-1}^{0} -x\ dx + \int_0^2 x\ dx\right)$

$Ex)\ \int_0^{3\pi/2} |\sin x|\ dx = \int_0^{\pi} \sin x\ dx + \int_{\pi}^{3\pi/2} -\sin x\ dx$

***Norm of P** = $\|P\|$*

$\lim \|p\| \to 0 \sum_{x}^{n} f(x)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\ dx$

$Ex) \lim \|p\| \to 0 \sum_{\beta}^{n} x_\beta^3 + x_\beta \sin x_\beta\, \Delta x\ [0, \pi] \to \int_0^{\pi} (x^3 + x\sin x)\, dx$

Properties of Definite Integral

1. $\int_a^b C\ dx = C(b-a)$
2. $\int_a^b [f(x) \pm g(x)]\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$
3. $\int_a^b C\, f(x)\ dx = C\int_a^b f(x)\ dx$
4. $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
5. $\int_a^a f(x)\ dx = 0$
6. $\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx \qquad a \le c \le b$
7. $f(x) \ge 0, \qquad a \le x \le b \qquad \int_a^b f(x)\ dx \ge 0$
8. $f(x) \ge g(x), \qquad a \le x \le b \qquad \int_a^b f(x)\ dx \ge \int_a^b g(x)\ dx$
9. $m \le f(x) \le M, \qquad a \le x \le b \qquad m(b-a) \le \int_a^b f(x)\ dx \le M(b-a)$

The Fundamental Theorem of Calculus

Suppose f is continuous on [a, b]

1. If $g(x) = \int_a^x f(t)\ dt,$ then $g'(x) = f(x)$

$Ex)\ S(x) = \int_0^x \sin(\pi\ t^2/2)\ dt \quad \to \quad S'(x) = \sin(\pi\ t^2/2)$

2. $\int_a^b f(x)\ dx = \int_a^b F'(x)\ dx = F(b) - F(a),$

   where F is any antiderivative of f, that is, $F' = f$.

The Substitution Rule of Definite Integral

$\int_a^b \boldsymbol{f(g(x))\ \ g'(x)dx} = \int_{g(a)}^{g(b)} \boldsymbol{f(u)du}$

1. $g(x) = u \quad g'(x) = \frac{du}{dx} \quad g'(x)dx = du$
2. change interval $[a,b] \to [g(a), g(b)] \qquad \int_a^b (\ \ )\ dx \to \int_{g(a)}^{g(b)} (\ \ )\ du$
3. Don't change $u \to g(x)$ from $\int_{g(a)}^{g(b)} f(u)du$

$Ex)\ \int_0^4 \sqrt{2x+1}\ dx \quad (2x+1 = u, 2 = \frac{du}{dx}, dx = \frac{1}{2}du \to\ g(0) = 1, g(4) = 9)$

$= \int_1^9 u^{1/2}\ \frac{1}{2}du = \frac{1}{2}\int_1^9 u^{1/2}\, du = \frac{1}{2}\left[\frac{2}{3} \cdot u^{3/2}\right]_1^9 = \frac{1}{2}\left(9^{3/2} - 1^{3/2}\right) = \frac{26}{3}$

$Ex)\ \int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\ dx \qquad (\ln x = u, \frac{1}{x} = \frac{du}{dx}, , \frac{1}{x}dx = du, \to \ln e^4 = 4, \ln e = 1)$

$= \int_1^4 u^{-1/2}\ \frac{1}{x}dx = \int_1^4 u^{-1/2}\ du = \left[2u^{1/2}\right]_1^4 = 2(\sqrt{4} - \sqrt{1}) = 2 \cdot (2-1) = 2$

Integrals of Symmetric functions suppose f(x) is continuous on [-a, a]

1. If $f(x)$is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx$
2. . If $f(x)$is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\ dx = 0$

Areas between Curves

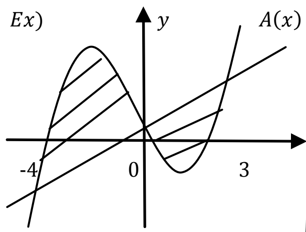
Suppose $f(x)$ and $g(x)$ are *continuous* and $\boldsymbol{f(x) \geq g(x)}$ for all x in [a,b]

$$\boldsymbol{A = \int_a^b [f(x) - g(x)]\ dx}$$

1. to find [a, b] ; when $f(x) = g(x) = 0$, x values are $[a, b]$
2. which is $f(x)$ or $g(x)$; test any No. between [a, b]
then the bigger func. is $f(x)$ and the other one is $g(x)$

$$A = \int_c^d [f(y) - g(y)]\ dy \qquad f(y) \geq g(y) \text{ for all y in } [c, d]$$

$Ex)$



$A(x) = y = x^3 + x^2 + 6x$

$B(x) = y = 6x$

Find the area of the shaded region.

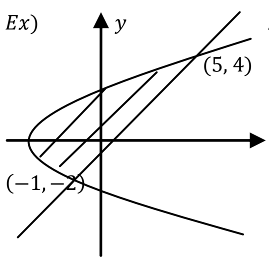
x on $[-4, 0]$ $\quad A(x) > B(x)$

on $[0, 3]$ $\quad B(x) > A(x)$

$$\int_{-4}^{0} [A(x) - B(x)]\ dx + \int_0^3 [B(x) - A(x)]\ dx$$

$$\therefore \int_{-4}^{0} [(x^3 + x^2 + 6x) - (6x)]\ dx + \int_0^3 [(6x) - (x^3 + x^2 + 6x)]\ dx$$

$Ex)$



$x_R = y + 1$

$x_L = \frac{1}{2}y^2 - 3$

Find the area of the shaded region.

1. Find [c, d]= [-2, 4]
2. find $f(y)$ and $g(y)$- test any No. in [-2, 4]

$f(0) = 0 + 1 = 1, \qquad f(0) = \frac{1}{2} \cdot 0 - 3 = -3$

then $x_R > x_L$

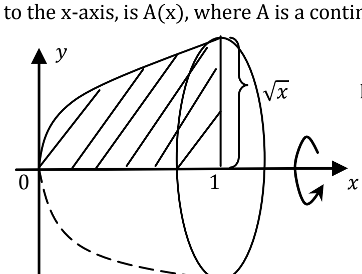
3. $\int_{-2}^{4} x_R - x_L\ dy = \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) dy = \int_{-2}^{4} \left(-\frac{1}{2}y^2 + y + 4\right)\ dy$

$$= \left[-\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y\right]_{-2}^{4} = \frac{64}{6} + 8 + 16 - \left(\frac{4}{3} + 2 - 8\right) = 18$$

$$\textbf{Volume of S} = \boldsymbol{V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\ dx = \int_a^b \pi\,(f(x))^2\ dx}$$

Let S be a solid that lies between x=a and x=b.

If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is A(x), where A is a continuous func.



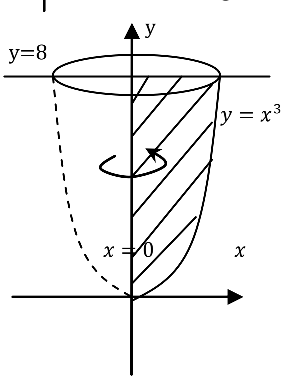
$Ex)\ y = \sqrt{x}\ \ [0, 1]$

Find the volume of the solid obtained by rotating about x − axis

$$V = \int_0^1 A(x)\ dx = \int_0^1 \pi\big(f(x)\big)^2\ dx$$

$$= \int_0^1 \pi\big(\sqrt{x}\big)^2\ dx = \pi \int_0^1 x\ dx = \frac{\pi}{2}$$

$*$ radius $= y = \sqrt{x}$, height $= [0, 1]$



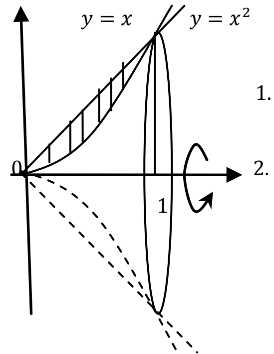
$Ex)\ y = x^3$ bounded by $y = 8$ & $x = 0$

Find the volume of the solid obtained by rotating about y − axis

$y = x^3 \rightarrow x = \sqrt[3]{y}$

$$V = \int_0^8 \pi\, y^{2/3}\ dy = \pi \left[\frac{3}{5} y^{5/3}\right]_0^8 = \frac{96\pi}{5}$$

$*$ **Two functions' case**; $\boldsymbol{A(x) = \pi(outer\ func.)^2 - \pi(inner\ func.)^2}$



$Ex)$ Find the volume of enclosed by curves $y = x$ & $y = x^2$ is rotating about the $x - axis$

1. Find [a, b] $\ y = x = x^2,\ x - x^2 = x(1 - x) = 0$
$x = 0$ or $1 \quad \therefore [a, b] = [0, 1]$
2. Find $f(x) \geq g(x),\quad f(x) = x,\ g(x) = x^2$
3. $\int_0^1 [\pi x^2 - \pi(x^2)^2]\ dx = \int_0^1 \pi(x^2 - x^4)\ dx =$

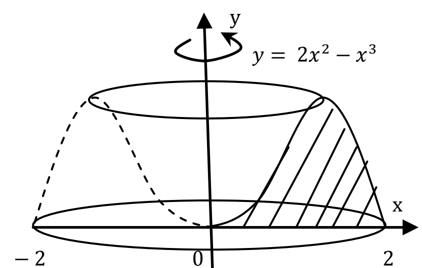
$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left[\left(\frac{1}{3} - \frac{1}{5}\right) - 0\right] = \pi\frac{2\pi}{15}$$

Volumes by Cylindrical Shells

$$\boldsymbol{V = \int_a^b (2\pi x)\cdot\big(f(x)\big)\cdot dx} = \int_a^b (circumference)\cdot(height)\cdot(thickness)$$

The volume of solid obtained by rotating about $y - axis$ the region under the curve $y = f(x)$ from a to b, where $0 \leq a < b$

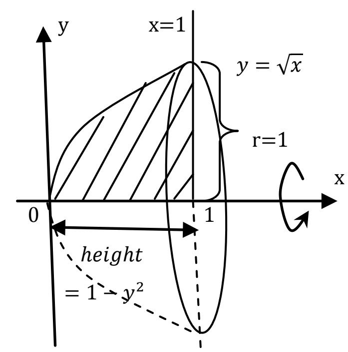
$Ex)$ Find the volume of the solid obtained by totating about the $y - axis$ the region bounded by $y = 2x^2 - x^3$ and $y = 0$



1. Find [a, b] ; the radius
$2x^2 - x^3 = x^2(2 - x) = 0$
$x = 0$ or $2 \rightarrow [0, 2]$
2. Find the Circumference
It's about $y - axis \rightarrow 2\pi x$
3. Find the height= $f(x)$

$$\therefore \int_0^2 (2\pi x)(2x^2 - x^3)\ dx$$

$Ex)$ Find the volume of the solid obtained by totating about the $x - axis$ the region bounded by $y = \sqrt{x}\ \ [0, 1]$



1. Find [a, b] ; the radius [0, 1]
2. Find the Circumference
It's about $x - axis \rightarrow 2\pi y$
3. Find the height= $f(y)$

Outer func. ; $x = 1$

Inner func. ; $y = \sqrt{x} \rightarrow x = y^2$

Outer func. −Inner func. $= 1 - y^2$

$$\therefore \int_0^1 (2\pi y)(1 - y^2)\ dy$$

$$= 2\pi \int_0^1 (y - y^3)\ dy = 2\pi\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = 2\pi\left[\left(\frac{1}{2} - \frac{1}{4}\right) - 0\right] = \frac{\pi}{2}$$

Average Values of Func.

If f is continuous on [a, b], then there exists a number c in [a, b] such that

$$\boldsymbol{f(c) = f_{ave} = \frac{1}{b - a}\int_a^b f(x)\ dx}$$

that is, $\int_a^b f(x)\ dx = f(c)(b - a)$

Ex) Find the average value of the $f(c) = 1 + x^2$ on $[-1, 2]$

$$f(c) = f_{ave} = \frac{1}{2 - (-1)}\int_{-1}^{2} (1 + x^2)\ dx = \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2} = 2$$

Work Problems

$velocity = v = f'(x) = \frac{ds}{dt}$

$acceleration = a = \frac{dv}{dt} = \frac{d}{dt}\frac{ds}{dt} = \frac{d^2 s}{dt^2}$

$\boldsymbol{F}orce = m \cdot a = \boldsymbol{m} \cdot \boldsymbol{\frac{d^2 s}{dt^2}} = m \cdot f''(x)$

$\boldsymbol{W}ork = \boldsymbol{F}orce \cdot \boldsymbol{d}isance = \boldsymbol{W} = \boldsymbol{F} \cdot \boldsymbol{d}$

$** \ \boldsymbol{W}'s\ unit\ is\ a\ newton - meter, which\ is\ called\ a\ joule\ (\boldsymbol{J}\,)$

***The Work done in moving the object from a to b**

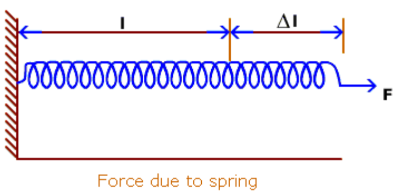
$\boldsymbol{W}ork \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$

$\boldsymbol{W}ork = \lim_{n\to\infty} \sum_{i=1}^{n} \boldsymbol{f}(\boldsymbol{x}_i^*)\Delta \boldsymbol{x} = \int_a^b \boldsymbol{f}(\boldsymbol{x})\ \boldsymbol{dx}$

Hooke's Law $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{k} \cdot \boldsymbol{x}$

$*$ $\boldsymbol{k}$ is a positive constant (called the **spring constant**)

a natural length



a stretched length

Ex) A force of $40N$ is required to hold a spring that has been stretched from its natural length of 10 cm to $15cm$.

a) Find the spring's force constant.

b)How much work is done in stretching the spring from $15cm$ to $18cm$?

a) the spring's force constant $= k$

1. Find it $from\ f(x) = k \cdot x$
2. Change the unit to m(Meter)
3. $40N = k \cdot (0.15m - 0.10m) = k \cdot 0.05m$
4. $k = \frac{40N}{0.05m} = 800\ N/m$

b) How much work is done $\rightarrow Work\ (J)$?

1. $f(x) = ?\ \ f(x) = k \cdot x = 800x$
2. [a, b]=? [0.05, 0.08]
3. $\int_{0.05}^{0.08} 800x\, dx = 800 \left[\frac{x^2}{2}\right]_{0.05}^{0.08} = 400[(0.08)^2 - (0.05)^2] = 1.56\, J$

3. Higher-order Derivatives

Definitions and properties

Second derivative

$$f'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) - \frac{d^2 y}{dx^2}$$

Higher-Order derivative

$$f^{(n)} = \left(f^{(n-1)}\right)'$$

$$(f+g)^{(n)} = f^{(n)} + g^{(n)}$$

$$(f-g)^{(n)} = f^{(n)} - g^{(n)}$$

Leibniz's Formulas

$$(f \cdot g)'' = f'' \cdot g + 2f' \cdot g' + f.g''$$

$$(f \cdot g)''' = f''' \cdot g + 3f'' \cdot g' + 3f' \cdot g'' + f \cdot g'''$$

$$(f \cdot g)^{(n)} = f^{(n)} g + nf^{(n-1)} g + \frac{n(n-1)}{1 \cdot 2} f^{(n-2)} g'' + \ldots + fg^{(n)}$$

Important Formulas

$$\left(x^m\right)^{(n)} = \frac{m!}{(m-n)!} x^{m-n}$$

$$\left(x^n\right)^{(n)} = n!$$

$$(\log_a x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n \cdot \ln a}$$

$$(\ln x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$\left(a^x\right)^{(n)} = a^x \ln^n a$$

$$\left(e^x\right)^{(n)} = e^x$$

$$\left(a^{mx}\right)^{(n)} = m^n a^{mx} \ln^n a$$

$$(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right)$$

$$(\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$$